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The maximal range problem for a quasidisk

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Abstract

Let $G\subset\mathbb{C}$ and $D\subset\mathbb{C}$ be simply connected domains such that $0\in G\cap D$. Denote by $\mathbb{P}_n$, $n\in\mathbb{N}:=\{1,2,\dots\}$, the set of all complex polynomials of degree at most n. Let

$$\mathbb{P}_n(G,D):=\{p\in\mathbb{P}_n\colon\ p(0)=0, p(G)\subset D\}.$$

Our main purpose is to find how large, i.e., how close to D, the "maximal polynomial range"

$$D_n(G):=\bigcup_{p\in\mathbb{P}_n(G,D)} p(G)$$

can be. We consider G to be a quasidisk and D to be an arbitrary domain whose boundary consists of more than two points.

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1. Introduction and main result

The notion of maximal ranges of polynomial spaces in the unit disk $\mathbb{D}:=\{w\colon |w|<1\}$ has been introduced in [5] and studied in [2,4,6,7,11,12] (for the survey of the various aspects of this notion, see [3]). This idea led to a unified approach to different inequalities for polynomials with constrains to their images of $\mathbb{D}$.

In this paper we generalize this concept by considering a quasidisk $G\subset\mathbb{C}$ (see [13]) instead of $\mathbb{D}$.

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Let $G\subset\mathbb{C}$ and $D\subset\mathbb{C}$ be simply connected domains such that $0\in G\cap D$. Denote by $\mathbb{P}_n,\ n\in\mathbb{N}:=\{1,2,\dots\}$, the set of all complex polynomials of degree at most n. Let

$$\mathbb{P}_n(G,D):=\{p\in\mathbb{P}_n:\ p(0)=0, p(G)\subset D\}.$$

Our main purpose is to find how large, i.e., how close to D, the "maximal polynomial range"

$$D_n(G):=\bigcup_{p\in\mathbb{P}_n(G,D)} p(G)$$

can be. We consider G to be a quasidisk, i.e., a finite domain bounded by a quasiconformal curve $L:=\partial G$. A geometric characterization of quasiconformal curves can be stated as follows (see [13, p. 100]): L is a quasiconformal curve iff there exists a constant $c>0$, depending only on L, such that for $z_1,z_2\in L$,

$$\min\{\operatorname{diam} L', \operatorname{diam} L''\}\leqslant c\,|z_1-z_2|, \tag{1.1}$$

where L' and L'' denote the two arcs that $L\setminus\{z_1,z_2\}$ consists of and $\operatorname{diam} E$ is the diameter of a set $E\subset\mathbb{C}$. Thus, we exclude regions with cusps on the boundary.

Let the boundary $\Gamma=\partial D$ of D consists of more than two points, and let $\Omega:=\overline{\mathbb{C}}\setminus\overline{G}$ and $\Delta:=\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ denote, respectively, the exterior (in the extended complex plane $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$) of $\overline{G}$ and $\overline{\mathbb{D}}$. Next we introduce conformal mappings

$$\Phi:\Omega\to\Delta,\quad \Phi(\infty)=\infty,\quad \Phi'(\infty)>0,$$

$$\phi:G\to\mathbb{D},\quad \phi(0)=0,\quad \phi'(0)>0,$$

$$\tilde{\phi}:D\to\mathbb{D},\quad \tilde{\phi}(0)=0,\quad \tilde{\phi}'(0)>0,$$

and the inverse mappings $\Psi:=\Phi^{-1},\psi:=\phi^{-1}$ and $\tilde{\psi}:=\tilde{\phi}^{-1}$. For the homeomorphism $g:=\phi\circ\Psi$ of the unit circle $\mathbb{T}:=\{w:\ |w|=1\}$ onto itself we define

$$\alpha(\delta)=\alpha(\delta,G):=\min_{w\in\mathbb{T}}|g(w)-g(we^{i\delta})|,\quad 0<\delta\leqslant\pi.$$

The function $\alpha(\delta)$ depends on geometric properties of G; we mention two examples. In the case $G=\mathbb{D}$ we have

$$\frac{2\delta}{\pi}\leqslant\alpha(\delta,\mathbb{D})=2\sin\frac{\delta}{2}\leqslant\delta. \tag{1.2}$$

The case of a piecewise smooth curve L is less trivial. Following [14], a smooth Jordan curve L is called Dini-smooth if the angle $\beta(s)$ of the tangent, considered as a function of the arc length s, satisfies

$$|\beta(s_2)-\beta(s_1)|<h(s_2-s_1)\quad (s_1<s_2),$$

where $h(x)$ is an increasing function for which

$$\int_0^1\frac{h(x)}{x}\,dx<\infty.$$

We call a Jordan arc Dini-smooth if it is a subarc of some Dini-smooth curve.

It follows from well-known distortion properties of conformal mappings ϕ and Ψ (see [14, Chapter 3]) that if L consists of a finite number m of Dini-smooth arcs which meet under inner angles $\beta_j\pi,\ 0<\beta_j<2,\ j=1,\dots,m$ with respect to G, then

$$\frac{1}{c_1}\,\delta^{\beta}\leqslant\alpha(\delta)\leqslant c_1\,\delta^{\beta}, \tag{1.3}$$

where

$$\beta=\beta(G):=\max\left\{\frac{2-\min_{1\leqslant j\leqslant m}\beta_j}{\min_{1\leqslant j\leqslant m}\beta_j},1\right\} \tag{1.4}$$

and $c_1=c_1(G)>1$ is a constant.

Note that according to (1.4), $\beta\geqslant 1$.

Following [2] we relate $D_n(G)$ to the images $\tilde{\psi}_s(\mathbb{D})$, where

$$\tilde{\psi}_s(z):=\tilde{\psi}((1-s)z),\quad 0<s<1,\ z\in\mathbb{D}.$$

Theorem. *There exist constants $c_0>0$ and $n_0\in\mathbb{N}$ that depend only on G, such that*

$$\tilde{\psi}_{c_0\alpha(1/n)}(\mathbb{D})\subset D_n(G),\quad n>n_0. \tag{1.5}$$

If we let $G=\mathbb{D}$ in the theorem above and take into account (1.2), then we obtain the second part of [2, Theorem 1] which is sharp for all unbounded domains [2, Theorem 2] and some bounded domains [3, Theorem 14].

Observe that for any quasidisk G,

$$\alpha(\delta)\leqslant c_2\delta,\quad c_2=c_2(G)>0,\ 0<\delta<\pi.$$

For domains with piecewise Dini-smooth boundary the above property follows from (1.3) and (1.4). For general quasidisks this property can be easily proved by applying the general distortion theory for conformal mappings, see [1]. From the comparison of the theorem above and [2, Theorem 1] we see the surprising fact that the estimate of the size of $D_n(G)$, measured in terms of function $\tilde{\psi}$, is worst for the unit disk. The proof of the theorem above is a straightforward combination of the following two lemmas. Let

$$L_\delta:=\{z:\ |\Phi(z)|=1+\delta\},\quad \delta>0.$$

Denote by ϕ_δ the conformal mapping of the domain $G_\delta:=\operatorname{int}L_\delta$, onto $\mathbb{D}$ normalized by the conditions $\phi_\delta(0)=0$, $\phi_\delta'(0)>0$. Here and in the sequel "int" means the interior of the indicated Jordan curve. For $\delta>0$ and real θ set

$$f_{\delta,\theta}(z):=\tilde{\psi}[e^{i\theta}\phi_{2\delta}(z)],\quad z\in G_{2\delta}.$$

Lemma 1. *There exist constants $c_3>0$ and $n_1\in\mathbb{N}$ with the following property: For any $n\in\mathbb{N},\ n>n_1$, $\delta=\delta_n:=c_3/n$ and $z\in\overline{G}$ there is a polynomial $p_{\delta,\theta,n,z}\in\mathbb{P}_n(G,D)$*

such that

$$f_{\delta,\theta}(z)\in p_{\delta,\theta,n,z}(G).$$

For the proof, see Section 3.

In order to state the second lemma we introduce a local version of the function $\alpha(\delta)$. For $0<\delta\leqslant\pi$, $z\in L=\partial G$ and $g:=\phi\circ\Psi$ set

$$\alpha_z(\delta):=|g(e^{i\delta}\phi(z))-g(\phi(z))|.$$

Lemma 2. *For any* $0<\delta\leqslant 1$ *and* $z\in L$ *the inequality*

$$1-|\phi_\delta(z)|\leqslant c_4\alpha_z(\delta) \tag{1.6}$$

holds with some constant $c_4=c_4(G)>0$.

For the proof, see Section 2.

Proof of Theorem. From Lemma 1 we conclude that for $n>n_1$,

$$D_n(G)\supset\bigcup_{0\leqslant\theta\leqslant 2\pi} f_{c_3/n,\theta}(\overline{G})=\tilde{\psi}\left(\left\{w:\ |w|\leqslant\max_{z\in L}|\phi_{2c_3/n}(z)|\right\}\right).$$

In view of Lemma 3 below and the reasoning in the beginning of Section 2 the function $\alpha(\delta)$ satisfies

$$\alpha(2\delta)\leqslant c_5\alpha(\delta),\quad \delta>0,$$

with some constant $c_5=c_5(G)>0$. Hence, Lemma 2 for sufficiently large n yields

$$D_n(G)\supset\tilde{\psi}\left(\left\{w:\ |w|\leqslant 1-c_4\min_{z\in L}\alpha_z\left(\frac{2c_3}{n}\right)\right\}\right)$$
$$\supset\tilde{\psi}\left(\left\{w:\ |w|\leqslant 1-c_0\,\alpha\left(\frac{1}{n}\right)\right\}\right).\qquad\square$$

We conclude this section with additional notations we use throughout the rest of the paper. Let $c,c_1,c_2,\ldots$ and $m,k,l,s\in\mathbb{N}$ be sufficiently large (>1) constants. Let also $\varepsilon,\varepsilon_1,\ldots$ be sufficiently small (<1) positive constants. The same symbol (e.g. c_1) may mean different constants in different relations.

We use for $a>0$ and $b>0$ order inequality $a\preccurlyeq b$ if $a\leqslant cb$. We also use $a\asymp b$ for $a\preccurlyeq b$ and $b\preccurlyeq a$ simultaneously.

For subsets $A,B\subset\mathbb{C}$ we set

$$d(A,B)=\operatorname{dist}(A,B):=\inf_{z\in A,\zeta\in B}|z-\zeta|.$$

2. Quasiconformal mappings

Let L be a K-quasiconformal curve (see [13]). It is well known that ϕ and Φ can be extended to K^2-quasiconformal homeomorphisms of $\overline{\mathbb{C}}$ onto itself. Slightly abusing the notation, we continue to write $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ for the latter and denote by $\phi^* : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the former. In view of Lemma 3 below we prefer to work with quasiconformal homeomorphisms of the complex plane that preserve the point at ∞. The mapping Φ satisfies this condition by definition. If $\phi^*(\infty) = \infty$, then for the inner conformal mapping ϕ we have the necessary quasiconformal extension $\phi := \phi^*$. If $\phi^*(\infty) = w_0 \neq \infty$ we modify ϕ^* in the following way.

The function $h_b(z) = h_b(x+iy) := x + iby$, where $1 < b < \infty$ is a parameter, maps the upper half plane, b-quasiconformally, onto itself. Combining this function with Möbius transformations we construct a $(|w_0|+1)/(|w_0|-1) =: K_1$-quasiconformal homeomorphism $\mu : \Delta \to \Delta$ with $\mu(w_0) = \infty$ and $\mu(w) = w$ on the unit circle $\mathbb{T}$. Hence, $\mu \circ \phi^*$ is the required $K_2 = K^2 K_1$-quasiconformal extension of ϕ. We denote $\mu \circ \phi^*$ also by ϕ. The inverse mappings $\psi := \phi^{-1}$ and $\Psi := \Phi^{-1}$ are also quasiconformal with coefficients of quasiconformality K_2 and K^2, respectively.

Next we recall one auxiliary fact from the theory of quasiconformal mappings.

Lemma 3 (Andrievskii et al. [1, p. 97]). *Suppose a function $w = F(\zeta)$ is a K-quasiconformal mapping of the extended complex plane onto itself, with $F(\infty) = \infty$. Assume also that $\zeta_j \in \mathbb{C}, w_j := F(\zeta_j), j = 1,2,3$. Then,*

(i) *the conditions $|\zeta_1 - \zeta_2| \leqslant c_1 |\zeta_1 - \zeta_3|$ and $|w_1 - w_2| \leqslant c_2 |w_1 - w_3|$ are equivalent, besides, the constants c_1 and c_2 are mutually dependent and also depend on K;*

(ii) *if $|\zeta_1 - \zeta_2| \leqslant c_1 |\zeta_1 - \zeta_3|$, then*

$$\varepsilon_1 \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{1/K} \leqslant \left| \frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2} \right| \leqslant c_3 \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K},$$

where $\varepsilon_1 = \varepsilon_1(c_1, K), c_3 = c_3(c_1, K)$.

As a first direct consequence of Lemma 3 we formulate the following statement. Let F be as in Lemma 3, and let $z_1, z_2, \zeta_1, \zeta_2 \in \mathbb{C}$ be such that

$$|\zeta_1 - \zeta_2| \asymp |z_1 - z_2| \preccurlyeq |z_1 - \zeta_1|.$$

Then,

$$\left| \frac{F(z_1) - F(\zeta_1)}{F(z_1) - F(z_2)} \right| \preccurlyeq \left| \frac{F(z_1) - F(\zeta_1)}{F(\zeta_1) - F(\zeta_2)} \right|^{K^2}. \tag{2.1}$$

Observe that the level curve L_δ is K^2-quasiconformal. This condition ensures that ϕ_δ can be extended quasiconformally to $\overline{\mathbb{C}}$ such that $\phi_\delta(\infty) = \infty$. We do this in the following way.

For $z \in \Omega_\delta := \overline{\mathbb{C}} \setminus \overline{G_\delta}$ we set

$$\phi_\delta^*(z) := \left[\overline{\phi_\delta\left(\Psi\left(\frac{(1+\delta)^2}{\overline{\Phi(z)}}\right)\right)} \right]^{-1}.$$

This function is a K^2-quasiconformal mapping of Ω_δ onto Δ, whose boundary values coincide with boundary values of ϕ_δ.

If $\Psi(0) =: \xi_0 = 0$, then $\phi_\delta^*(\infty) = \infty$ and we necessarily obtain the $K_1 = K^2$-quasiconformal extension of ϕ_δ by setting $\phi_\delta = \phi_\delta^*$ in Ω_δ.

If $\xi_0 \neq 0$ we modify ϕ_δ^* as follows. By Schwarz's lemma, applied to the function $\phi_\delta \circ \psi$, we obtain

$$\frac{1}{|\phi_\delta^*(\infty)|} = |\phi_\delta(\xi_0)| < |\phi(\xi_0)|.$$

Therefore, if we reason as in the beginning of this section we derive that there exists a quasiconformal mapping $\mu_\delta : \Delta \to \Delta$ with $\mu_\delta(w) = w$ on $\mathbb{T}$ and $\mu_\delta(\phi_\delta^*(\infty)) = \infty$, whose coefficient of quasiconformality is at most

$$\frac{|\phi_\delta^*(\infty)| + 1}{|\phi_\delta^*(\infty)| - 1} < \frac{1 + |\phi(\xi_0)|}{1 - |\phi(\xi_0)|}.$$

The function $\phi_\delta := \mu_\delta \circ \phi_\delta^*$ in Ω_δ is a required $K_1 = K_1(G, K)$-quasiconformal extension of ϕ_δ.

The quasiconformal extensions above show that to study the metric properties of Φ and ϕ_δ we can use Lemma 3. Moreover, all constants in this lemma depend only on G and K and they are independent of δ.

Let $S = S_\delta := \phi_\delta(L)$. For $t \in \overline{\mathbb{D}} \setminus \{0\}$ we set $t_{\mathbb{T}} := t/|t|$. Note that for any $\tau \in \mathbb{T}; t_1, t_2 \in S \cap [0, \tau]$ we have

$$|t_1 - \tau| \asymp |t_2 - \tau|. \tag{2.2}$$

Indeed, without loss of generality we may assume that $\tau = 1$. Since $J_\pm := [-1, 1] \cup \{\tau \colon |\tau| = 1, \pm \operatorname{Im} \tau > 0\}$ are K_0-quasiconformal (recall the Ahlfors criterium (1.1)), the curves $\Phi \circ \psi_\delta(J_\pm)$ are $K_0 K^2 K_1$-quasiconformal. Therefore, by (1.1) we have

$$|\Phi \circ \psi_\delta(t_1) - \Phi \circ \psi_\delta(\tau)| \asymp |\Phi \circ \psi_\delta(t_2) - \Phi \circ \psi_\delta(\tau)|,$$

from which (2.2) follows by virtue of Lemma 3.

For $t \in \overline{\mathbb{D}} \setminus \{0\}$, denote by t_S the point of $[0, t_{\mathbb{T}}] \cap S$ with the smallest modulus. The discussion above shows that

$$|\Phi \circ \psi_\delta(\tau_S) - \Phi \circ \psi_\delta(\tau)| \preccurlyeq \delta \leqslant |\xi - \Phi \circ \psi_\delta(\tau)|$$

holds for any $\xi \in \mathbb{T}$. Hence, setting ξ so that

$$|\phi_\delta \circ \Psi(\xi) - \tau| = d(\tau, S),$$

and making use of Lemma 3, we obtain

$$|\tau - \tau_S| \preccurlyeq d(\tau, S). \tag{2.3}$$

Furthermore, from Lemma 3, for $\tau\in\mathbb{T}$ and $\delta<1$ we obtain

$$\delta\leqslant 2\left|\frac{\Phi\circ\psi_\delta(\tau_S)-\Phi\circ\psi_\delta(\tau)}{\Phi\circ\psi_\delta(0)-\Phi\circ\psi_\delta(\tau)}\right|\preccurlyeq\left|\frac{\tau_S-\tau}{0-\tau}\right|^{\varepsilon_1}=|\tau-\tau_S|^{\varepsilon_1}. \tag{2.4}$$

Next, for any $\tau,w\in\mathbb{T}$ with $|\tau-w|\leqslant 2|\tau-\tau_S|$ we have

$$|w-w_S|\asymp|\tau-\tau_S|. \tag{2.5}$$

Indeed, setting

$$v:=\Phi\circ\psi_\delta(\tau),\quad v_S:=\Phi\circ\psi_\delta(\tau_S),$$

$$\eta:=\Phi\circ\psi_\delta(w),\quad \eta_S:=\Phi\circ\psi_\delta(w_S),$$

and using Lemma 3 we obtain

$$|v-\eta|\preccurlyeq\delta\asymp|v-v_S|\asymp|\eta-\eta_S|.$$

Furthermore, since

$$|v-v_S|\asymp|\eta-v_S|\asymp|\eta-\eta_S|,$$

an application of Lemma 3 gives

$$|\tau-\tau_S|\asymp|w-\tau_S|\asymp|w-w_S|.$$

Our next objective is to show that for any $\tau,w\in\mathbb{T}$ with $|\tau-w|\geqslant|\tau-\tau_S|$ the inequality

$$|w-w_S|\preccurlyeq|\tau-\tau_S|^{\varepsilon}|\tau-w|^{1-\varepsilon} \tag{2.6}$$

holds.

Indeed, by making use of Lemma 3 for points v,v_S,η,η_S, introduced above we have

$$|v-\eta|\succcurlyeq\delta\asymp|v-v_S|\asymp|\eta-\eta_S|.$$

Thus, by Lemma 3

$$\left|\frac{w-w_S}{w-\tau}\right|\preccurlyeq\left|\frac{\eta-\eta_S}{\eta-v}\right|^{\varepsilon_2}\asymp\left|\frac{v-v_S}{v-\eta}\right|^{\varepsilon_2}\preccurlyeq\left|\frac{\tau-\tau_S}{\tau-w}\right|^{\varepsilon_3},$$

which gives (2.6) with $\varepsilon:=\varepsilon_3$.

Proof of Lemma 2. Let $t:=\phi_\delta(z)$. We may assume that δ is sufficiently small, so that the constructions below are well defined. For simplicity we also assume that $t_{\mathbb{T}}=1, t=t_S$ (cf. (2.2)). Set $h:=1-t$.

According to (2.3), (2.5) and (2.6) there exist constants ε and c_1 such that the curve

$$\{re^{i\theta}:r=c_1h,|\theta|\leqslant h\}\cup\left\{re^{i\theta}:\ r=c_1h^{\varepsilon}|\theta|^{1-\varepsilon},h\leqslant|\theta|\leqslant\frac{\pi}{2}\right\}$$
$$\cup\left\{re^{i\theta}:\ r=c_1\left(\frac{\pi}{2}\right)^{1-\varepsilon}h^{\varepsilon},\frac{\pi}{2}\leqslant|\theta|\leqslant\pi\right\}$$

lies in the interior of S.

Set $\chi := \phi \circ \psi_\delta$, $t_1 := 1 - 4c_1 h$, $\chi(t) =: \xi$, $\chi(t_1) =: \xi_1$, $\psi_\delta(t_1) =: z_1$. Then Lemma 3 gives

$$|z - z_1| \asymp |z - \psi_\delta(1)| \asymp d(z, L_\delta) \asymp |z - \Psi(e^{i\delta}\Phi(z))|,$$

and therefore

$$|\xi - \xi_1| \asymp \alpha_z(\delta). \tag{2.7}$$

Denote by $\Gamma_0 = \Gamma_0(t, t_1, \text{int } S)$ the family of all cross-cuts of int S which separate points t and t_1 from 0 in int S.

Observe that inequality (1.6) follows from the estimate

$$m(\Gamma_0) \leqslant \frac{1}{\pi} \log \frac{1}{h} + c_2. \tag{2.8}$$

Indeed, taking into account the conformal invariance of moduli (see [13]), [2, Lemma 8], (2.7) and (2.8), we have

$$\frac{1}{\pi} \log \frac{1}{\alpha_z(\delta)} - c_3 \leqslant \frac{1}{\pi} \log \frac{1}{|\xi - \xi_1|} \leqslant m(\chi(\Gamma_0))$$
$$= m(\Gamma) \leqslant \frac{1}{\pi} \log \frac{1}{h} + c_2,$$

from which (1.6) follows.

In order to prove (2.8) we first simplify (geometrically) our reasoning by using the auxiliary Möbius transformation

$$v(\tau) := \frac{1 - \tau}{1 + \tau}.$$

Then if we set $v(t) =: u, v(t_1) =: u_1$, we see that

$$\frac{h}{2} \leqslant u \leqslant h,$$

$$2c_1 u \leqslant 2c_1 h \leqslant u_1 \leqslant 4c_1 h \leqslant 8c_1 u < 1.$$

Moreover, there exist constants ε_1 and c_4 such that the function

$$v(r) := \frac{\pi}{2} \max\left\{\varepsilon_1, 1 - c_4 \left(\frac{h}{r}\right)^{\varepsilon}\right\}$$

satisfies

$$\{re^{i\theta}\colon u_1 \leqslant r \leqslant 1, |\theta| \leqslant v(r)\} \subset v(\text{int } S).$$

Inequality (2.8) follows directly from the estimate

$$m(\Gamma_0) = m(\Gamma') \leqslant \frac{1}{\pi} \log \frac{1}{u} + c_5, \tag{2.9}$$

where $\Gamma' := v(\Gamma_0)$.

In order to prove (2.9) we consider the following functions:

$$\rho_1(z) = c_6, \quad |z| \leqslant 2, \quad \operatorname{Re} z \geqslant 0,$$

$$\rho_2(z) = \frac{c_7}{u}, \quad |z| \leqslant 2u_1, \quad \operatorname{Re} z \geqslant 0,$$

$$\rho_3(z) = \frac{1}{2v(|z|e^{-\pi})|z|}, \quad u_1 \leqslant |z| \leqslant 1, \quad \operatorname{Re} z \geqslant 0.$$

We extend these functions to be 0 in the rest of the complex plane, and we set

$$\rho(z) := \max\{\rho_1(z), \rho_2(z), \rho_3(z)\}, \quad z \in \mathbb{C}.$$

We claim that constants c_6 and c_7 can be chosen so that, for any $\gamma \in \Gamma'$, the inequality

$$\int_\gamma \rho(z)|dz| \geqslant 1 \tag{2.10}$$

holds.

To verify (2.10) we consider three particular cases. Let $|\gamma|$ denote the length of γ. Let $\gamma \cap \mathbb{T} \neq \emptyset$. Then $|\gamma| \geqslant \varepsilon_2$. If we take $c_6 = 1 + 1/\varepsilon_2$, then

$$\int_\gamma \rho_1(z)|dz| \geqslant 1. \tag{2.11}$$

Let $\gamma \cap \{z\colon |z| = |u_1|\} \neq \emptyset$. Then $|\gamma| \geqslant \varepsilon_3 u$. Taking $c_7 = 1 + 1/\varepsilon_3$, we have

$$\int_\gamma \rho_2(z)|dz| \geqslant 1. \tag{2.12}$$

Let $\gamma \subset \{z\colon u_1 < |z| < 1\}$, and

$$r_{\max} := \sup_{z \in \gamma} |z|, \quad r_{\min} := \inf_{z \in \gamma} |z|.$$

If $r_{\max} \geqslant e^\pi r_{\min}$, then there exists a γ_1 with $\gamma_1 \subset \gamma$, which connects the circular boundary parts of the annulus

$$\{z\colon r_{\min} < |z| < r_{\max}\}$$

and, consequently,

$$\int_\gamma \rho_3(z)|dz| \geqslant \frac{1}{\pi}\left|\int_{\gamma_1} \frac{dz}{z}\right| \geqslant \frac{1}{\pi} \log \frac{r_{\max}}{r_{\min}} \geqslant 1. \tag{2.13}$$

If $r_{\max} < e^\pi r_{\min}$, then

$$\int_\gamma \rho_3(z)|dz| \geqslant \frac{1}{2v(r_{\min})}\left|\int_\gamma \frac{dz}{z}\right| \geqslant 1. \tag{2.14}$$

Comparing (2.11)–(2.14), we obtain (2.10).

By the definition of the module (see [13]) we get

$$m(\Gamma') \leqslant \int \rho^2(z)\ dm(z) \leqslant \sum_{j=1}^{3} \int \rho_j^2(z)\ dm(z), \tag{2.15}$$

where $dm(z)$ stands for the 2-dimensional Lebesgue measure.

An easy computation shows that

$$\sum_{j=1}^{2} \int \rho_j^2(z)\ dm(z) \leqslant c_8. \tag{2.16}$$

To estimate the third integral in (2.15) we write

$$\begin{aligned}\int \rho_3^2(z)\ dm(z) &= \int_{u_1}^{1} \frac{dr}{2v(e^{-\pi}r)r} = \frac{1}{\pi}\int_{u_1}^{1}\frac{dr}{r} + \frac{1}{2\pi}\int_{u_1}^{1}\frac{\pi - 2v(e^{-\pi}r)}{v(e^{-\pi}r)r}\ dr \\ &\leqslant \frac{1}{\pi}\log\frac{1}{u_1} + \frac{c_4 e^{\pi\varepsilon}h^{\varepsilon}}{\varepsilon_1 \pi}\int_{u_1}^{1}\frac{dr}{r^{1+\varepsilon}} \leqslant \frac{1}{\pi}\log\frac{1}{u} + c_9. \end{aligned} \tag{2.17}$$

The required inequality (2.9) follows from the combination of (2.15)–(2.17). □

3. Polynomial approximation on a quasidisk

Let f be analytic in $G_{2\delta}$, $0<\delta<1$. We describe a construction of polynomials approximating f in $\overline{G_\delta}$ (for more details, see [1]). Consider an antiderivative of f, i.e., the function

$$F(\zeta) := \int_{\gamma(\zeta)} f(\xi)\, d\xi, \quad \zeta \in \overline{G_\delta},$$

where $\gamma(\zeta) \subset \overline{G_\delta}$ is an arbitrary rectifiable arc joining 0 and ζ. We are going to describe the structural properties of f in terms of its local modulus of continuity

$$\omega_{f,z,\overline{G_\delta}}(h) := \max_{\zeta\in\overline{G_\delta}, |z-\zeta|\leqslant h} |f(z) - f(\zeta)|, \quad z \in \overline{G_\delta}, \ h>0.$$

We will need the continuous extension of F into the complex plane which preserves its smoothness properties. The corresponding construction, proposed by Dyn'kin [9,10], is based on the Whitney unity partition (see [15]) and properties of the local modulus of continuity of F. Note that for any $z, \zeta \in \overline{G_\delta}$ there exists an arc $\gamma \subset \overline{G_\delta}$, joining z and ζ, whose length $|\gamma|$ satisfies the condition (cf. [1, p. 24])

$$|\gamma| \preccurlyeq |z-\zeta|. \tag{3.1}$$

A slight modification of the reasoning in [9,10,15], as well as an application of (3.1), gives the following result (cf. [1, pp. 13–15]).

Lemma 4. *The function F can be continuously extended from $\overline{G_\delta}$ to $\mathbb{C}$ (we preserve the notation F for the extension) such that:*

(i) $F(z) = 0$ *for z with $d(z, \overline{G_\delta}) \geqslant 3$, i.e., F has compact support*;

(ii) *for* $z \in \mathbb{C} \setminus \overline{G_\delta}$,

$$\left| \frac{\partial F(z)}{\partial \bar{z}} \right| \preccurlyeq \omega_{f, z_\delta, \overline{G_\delta}}(23 d(z, \overline{G_\delta})),$$

where $z_\delta \in L_\delta$ *is an arbitrary point among the ones that are closest to* z;

(iii) *for* $z \in G_\delta$,

$$f(z) \coloneqq -\frac{1}{\pi} \int_{\Omega_\delta} \frac{\partial F(\zeta)}{\partial \bar{\zeta}} \frac{dm(\zeta)}{(\zeta - z)^2}.$$

In order to approximate the Cauchy kernel $1/(\zeta - z)$, $z \in \overline{G}$, $\zeta \in \overline{\Omega}$, by polynomial kernels of the form

$$K_n(\zeta, z) = \sum_{j=0}^{n} a_j(\zeta) z^j, \tag{3.2}$$

we use the functions $K_{r,m,k,n}(\zeta, z)$ introduced by Dzjadyk (see [8, Chapter 9] or [1, Chapter 3]). Taking them as a basis for our discussion we mention the following result. Let

$$\tilde{z} = \tilde{z}_{1/n} \coloneqq \Psi[(1 + 1/n)\Phi(z)], \quad z \in \overline{\Omega}.$$

Lemma 5. *Let* $k, m, s \in \mathbb{N}$. *Then for any* $n \in \mathbb{N}$ *there exists a polynomial kernel of the form* (3.2) *such that the following relation holds for* $l = 0, 1, \ldots, s$, $z \in \overline{G}$ *and* $\zeta \in \overline{\Omega}$ *with* $d(\zeta, L) \leqslant 3$:

$$\left| \frac{\partial^l}{\partial z^l} \left(\frac{1}{\zeta - z} - K_n(\zeta, z) \right) \right| \preccurlyeq \frac{1}{|\zeta - z|^{l+1}} \left| \frac{\tilde{\zeta} - \zeta}{\tilde{\zeta} - z} \right|^k$$
$$\preccurlyeq \frac{1}{|\zeta - z|^{l+1}} \left(\frac{|z_0 - \tilde{z}_0|}{|\zeta - z| + |z_0 - \tilde{z}_0|} \right)^m,$$

where $z_0 \in L$ *is an arbitrary point on* L *among the ones that are the closest to* z.

We now turn to the

Proof of Lemma 1. Let $f \coloneqq f_{\delta,\theta}$. According to Lemma 4

$$f(z) = \int_{\Omega_\delta} \frac{\lambda(\zeta)}{(\zeta - z)^2} dm(\zeta), \quad z \in \overline{G},$$

where

$$|\lambda(\zeta)| = \left| -\frac{1}{\pi} \frac{\partial F(\zeta)}{\partial \bar{\zeta}} \right| \preccurlyeq \omega_{f, \zeta_\delta, \overline{G_\delta}}(23|\zeta - \zeta_\delta|).$$

We define an auxiliary polynomial $t_n \in \mathbb{P}_{n-1}$ as follows

$$t_n(z) := \int_{\Omega_\delta} \lambda(\zeta) \frac{\partial}{\partial z} K_n(\zeta, z)\; dm(\zeta), \quad z \in \overline{G},$$

where $K_n(\zeta, z)$ is a polynomial kernel from Lemma 5. By this lemma for $z \in \overline{G}$ we have

$$|f(z) - t_n(z)| \preccurlyeq \int_{\Omega_\delta} \frac{\omega_{f,z,\overline{G_\delta}}(25|\zeta - z|)}{|\zeta - z|^2} \left(\frac{|z_0 - \tilde{z}_0|}{|\zeta - z| + |z_0 - \tilde{z}_0|} \right)^m dm(\zeta). \tag{3.3}$$

Next we estimate $\omega_{f,z,\overline{G_\delta}}(h), z \in \overline{G}$, for $d(z, L_\delta) < h \leqslant$ diam $\overline{G_\delta}$.

Let $z \in \overline{G}$ and $\zeta \in \overline{G}_\delta$ be such that $|z - \zeta| \leqslant h$, and let $z' := e^{i\theta}\phi_{2\delta}(z)$, $\zeta' := e^{i\theta}\phi_{2\delta}(\zeta)$. By Andrievskii and Ruscheweyh [2, (7)]

$$\frac{|f(z) - f(\zeta)|}{d(f(z), \Gamma)} \leqslant \left(1 + 2 \frac{|z' - \zeta'|(|z' - \zeta'| + |1 - z'\bar{\zeta}'|)}{(1 - |z'|^2)(1 - |\zeta'|^2)} \right)^2.$$

Since

$$|1 - z'\bar{\zeta}'| \leqslant |1 - \zeta'\bar{\zeta}'| + |\zeta'\bar{\zeta}' - z'\bar{\zeta}'| \leqslant 1 - |\zeta'|^2 + |z' - \zeta'|,$$

it immediately follows that

$$\frac{|f(z) - f(\zeta)|}{d(f(z), \Gamma)} \leqslant \left(1 + 4 \frac{|z' - \zeta'|^2}{(1 - |z'|)(1 - |\zeta'|)} + 2 \frac{|z' - \zeta'|}{1 - |z'|} \right)^2.$$

Taking into account the fact that $d(\zeta, L_{2\delta}) \preccurlyeq |\zeta - z|$ and $d(z, L_{2\delta}) \preccurlyeq |\zeta - z|$ by using Lemma 3 we obtain $1 - |\zeta'| \preccurlyeq |\zeta' - z'|$ and $1 - |z'| \preccurlyeq |\zeta' - z'|$. Therefore,

$$\frac{|f(z) - f(\zeta)|}{d(f(z), \Gamma)} \preccurlyeq \left(\frac{|z' - \zeta'|}{1 - |z'|} \right)^2 \left(\frac{|z' - \zeta'|}{1 - |\zeta'|} \right)^2.$$

Further we note that

$$\frac{|z' - \zeta'|}{1 - |z'|} \preccurlyeq \left(\frac{|\zeta - z|}{d(z, L_{2\delta})} \right)^{c_1},$$

and by (2.1)

$$\frac{|z' - \zeta'|}{1 - |\zeta'|} \preccurlyeq \left(\frac{|\zeta - z|}{d(\zeta, L_{2\delta})} \right)^{c_1} \preccurlyeq \sup_{\zeta \in L_\delta, |\zeta - z| \leqslant h} \left(\frac{|\zeta - z|}{d(\zeta, L_{2\delta})} \right)^{c_1} \preccurlyeq \left(\frac{|\zeta - z|}{d(z_0, L_{2\delta})} \right)^{c_2}.$$

These lead to

$$\omega_{f,z,\overline{G_\delta}}(h) \preccurlyeq d(f(z), \Gamma) \left(\frac{h}{d(z_0, L_{2\delta})} \right)^c, \tag{3.4}$$

from which, by (3.3) and (3.4) for $z\in\overline{G}$, we obtain

$$\frac{|f(z)-t_n(z)|}{d(f(z),\Gamma)}\preccurlyeq\frac{|z_0-\tilde{z}_0|^m}{d(z_0,L_\delta)^c}\int_{\Omega_\delta}\frac{dm(\zeta)}{|\zeta-z|^{m+2-c}}$$
$$\preccurlyeq\frac{|z_0-\tilde{z}_0|^m}{d(z_0,L_\delta)^c}[d(z,L_\delta)]^{c-m}\preccurlyeq\left(\frac{|z_0-\tilde{z}_0|}{d(z,L_\delta)}\right)^{m-c}.$$

Therefore, fixing arbitrary $k\in\mathbb{N}$ and taking m sufficiently large we have

$$|f(z)-t_n(z)|\preccurlyeq d(f(z),\Gamma)\left(\frac{|z_0-\tilde{z}_0|}{d(z,L_\delta)}\right)^k,\quad z\in\overline{G}. \tag{3.5}$$

Since by Lemma 3

$$\frac{|z_0-\tilde{z}_0|}{d(z,L_\delta)}\preccurlyeq\frac{|z_0-\tilde{z}_0|}{d(z_0,L_\delta)}\preccurlyeq\left(\frac{1}{n\delta}\right)^{\varepsilon},$$

for $\delta=c_3/n$ with sufficiently large c_3, we obtain

$$\frac{|f(z)-t_n(z)|}{d(f(z),\Gamma)}<\frac{1}{2}. \tag{3.6}$$

Denote by $y(\zeta)$, $\zeta\in\mathbb{C}$, a quasiconformal reflection with respect to L, i.e., an antiquasiconformal mapping $y:\mathbb{C}\to\mathbb{C}$ with the properties $y(y(z))=z$, $y(G)=\Omega$, $y(\Omega)=G$ that keeps the points of L invariant (see [1,13]). For $\zeta\in\overline{G}$ we set

$$\tilde{\zeta}=\tilde{\zeta}_{1/n}:=\Psi\left(\left(1+\frac{1}{n}\right)\Phi(y(\zeta))\right).$$

Since for $z,\zeta\in\overline{G}, z\neq\zeta$, a straightforward induction on m gives

$$\frac{1}{\zeta-z}=\sum_{j=1}^{m}\frac{(\tilde{\zeta}-\zeta)^{j-1}}{(\tilde{\zeta}-z)^j}+\frac{(\tilde{\zeta}-\zeta)^m}{(\zeta-z)(\tilde{\zeta}-z)^m},$$

the polynomial (in z)

$$Q_n(\zeta,z):=\sum_{j=1}^{m}\frac{(\tilde{\zeta}-\zeta)^{j-1}}{(j-1)!}\frac{\partial^{j-1}}{\partial z^{j-1}}K_n(\tilde{\zeta},z)$$

satisfies, for $z\in L$, the following inequalities:

$$\left|\frac{1}{\zeta-z}-Q_n(\zeta,z)\right|\preccurlyeq\sum_{j=1}^{m}\frac{|\tilde{\zeta}-\zeta|^{j-1}}{|\tilde{\zeta}-z|^j}\left|\frac{\zeta-\tilde{\zeta}}{\tilde{\zeta}-z}\right|^m+\frac{1}{|\zeta-z|}\left|\frac{\tilde{\zeta}-\zeta}{\tilde{\zeta}-z}\right|^m$$
$$\preccurlyeq\frac{1}{|\zeta-z|}\left|\frac{\tilde{\zeta}-\zeta}{\tilde{\zeta}-z}\right|^m.$$

Hence, the polynomial (in z)

$$V_n(\zeta,z):=1-(\zeta-z)Q_n(\zeta,z)$$

satisfies

$$|V_n(\zeta,z)| \preccurlyeq \left|\frac{\tilde{\zeta}-\zeta}{\tilde{\zeta}-z}\right|^m \quad \zeta\in\overline{G},\ \ z\in L. \tag{3.7}$$

We also consider the polynomial (in z)

$$u_n(z)=u_n(\zeta,z):=\frac{z}{\zeta}[f(\zeta)-t_n(\zeta)]V_{n-1}(\zeta,z)+\frac{z-\zeta}{\zeta}t_n(0),$$

which has the properties

$$u_n(0)=-t_n(0), \quad u_n(\zeta)=f(\zeta)-t_n(\zeta). \tag{3.8}$$

Furthermore, for $z\in L$ and $\zeta\in\overline{G}, |\zeta|>\varepsilon_1$, (2.1), (3.5) and (3.7) imply that

$$\begin{aligned}|u_n(z)| &\preccurlyeq d(f(\zeta),\Gamma)\left|\frac{\tilde{\zeta}-\tilde{\tilde{\zeta}}}{\tilde{\zeta}-\zeta}\right|^m\left|\frac{\tilde{\zeta}-\zeta}{\tilde{\zeta}-z}\right|^m+\frac{1}{n^{k\varepsilon}}\\ &= d(f(\zeta),\Gamma)\left|\frac{\tilde{\zeta}-\tilde{\tilde{\zeta}}}{\tilde{\zeta}-z}\right|^m+\frac{1}{n^{k\varepsilon}}\\ &\preccurlyeq d(f(\zeta),\Gamma)\left|\frac{z-\tilde{z}}{z-\tilde{\zeta}}\right|^{m\varepsilon}+\frac{1}{n^{k\varepsilon}}\\ &\preccurlyeq d(f(\zeta),\Gamma)\left|\frac{z-\tilde{z}}{z-\tilde{\zeta}}\right|^{s}+\frac{1}{n^{l}}.\end{aligned}$$

We claim that for $\delta=c_3/n$ with sufficiently large c_3 the inequality

$$\frac{|u_n(z)|}{d(f(z),\Gamma)}<\frac{1}{2}, \quad z\in L \tag{3.9}$$

holds.

By using a variant of Löwner's inequality on the distance between level curves (see e.g. [1, p. 61]) and (2.4) we obtain

$$d(f(z),\Gamma)\succcurlyeq\delta^c, \quad z\in L.$$

Therefore, in order to establish (3.9) it is enough to show that the expression

$$B(\zeta,z):=\frac{d(f(\zeta),\Gamma)}{d(f(z),\Gamma)}\left|\frac{z-\tilde{z}}{z-\tilde{\zeta}}\right|^s$$

can be made arbitrarily small if c_3 is selected large enough. Assume first that $d(f(\zeta),\Gamma)\leqslant 2d(f(z),\Gamma)$. Then in view of Lemma 3, we obtain

$$B(\zeta,z)\leqslant 2\left|\frac{z-\tilde{z}}{z-\tilde{\zeta}}\right|^s \preccurlyeq \left|\frac{\Phi(z)-\Phi(\tilde{z})}{\Phi(z)-\Phi(\tilde{\zeta})}\right|^{s\varepsilon}\preccurlyeq c_3^{-s\varepsilon}$$

and (3.9) follows. Assume now that

$$d(f(\zeta),\Gamma)>2d(f(z),\Gamma).$$

Thus,

$$d(f(\zeta),\Gamma)\leqslant\frac{1}{2}|f(\zeta)-f(z)|.$$

By (3.4) we conclude that

$$\frac{|f(z)-f(\zeta)|}{d(f(z),\Gamma)}\preccurlyeq\left(\frac{|\zeta-z|}{d(z,L_\delta)}\right)^c$$

and for $s>c$

$$B(\zeta,z)\preccurlyeq\left(\frac{|\tilde{\zeta}-z|}{d(z,L_\delta)}\right)^c\left|\frac{z-\tilde{z}}{z-\tilde{\zeta}}\right|^s\preccurlyeq\left(\frac{|z-\tilde{z}|}{d(z,L_\delta)}\right)^{s-c}\preccurlyeq c_3^{c-s},$$

which also proves (3.9). Consider the polynomial

$$p_n(z):=t_n(z)+u_n(z).$$

According to (3.6), (3.8) and (3.9) it has the necessary properties, that is,

$$p_n(0)=0,\quad p_n(\zeta)=f(\zeta),\quad p_n(\overline{G})\subset D.\qquad\square$$

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References

[1] V.V. Andrievskii, V.I. Belyi, V.K. Dzjadyk, Conformal Invariants in Constructive Theory of Functions of Complex Variable, World Federation Publisher, Atlanta, GA, 1995.
[2] V.V. Andrievskii, St. Ruscheweyh, Maximal polynomial subordination to univalent functions in the unit disk, Constr. Approx. 10 (1994) 131–144.
[3] V.V. Andrievskii, St. Ruscheweyh, Complex polynomials and maximal range: background and applications, in: G.V. Milovanovic (Ed.), Recent Progress in Inequalities (Dedicated to Prof. Dragoslav S. Mitrinovic), Kluwer Academic Publishers, Dordrecht, Math. Appl. (Dordrecht) 430 (1998) 31–54.
[4] A. Córdova, St. Ruscheweyh, On maximal polynomial ranges in circular domains, Complex Variables 10 (1988) 295–309.
[5] A. Córdova, St. Ruscheweyh, On maximal ranges of polynomial spaces in the unit disk, Constr. Approx. 5 (1989) 309–328.
[6] A. Córdova, St. Ruscheweyh, On the maximal range problem for slit domains, in: Proceedings of a Conference, Valparaíso, 1989, Lecture Notes in Mathematics, Vol. 1435, Springer, Berlin, 1990, pp. 33–44.
[7] A. Córdova, St. Ruscheweyh, Subordination of polynomials, Rocky Mountain J. Math. 21 (1991) 159–170.
[8] V.K. Dzjadyk, Introduction to the Theory of Uniform Approximation of Functions by Polynomials, Nauka, Moskow, 1977 (in Russian).

[9] E.M. Dyn'kin, Uniform approximation of functions in Jordan domains, Sib. Mat. Zh. 18 (1977) 775–786 (in Russian).
[10] E.M. Dyn'kin, A constructive characterization of the Sobolev and Besov classes, Trudy Mat. Inst. Steklova 155 (1981) 41–76 (in Russian).
[11] R. Greiner, On a theorem of Andrievskii and Ruscheweyh, Proceedings of the Ashkelon Workshop in Complex Function Theory, 1996, Israel Mathematics Conference Proceedings, Vol. 11, Bar-Ilan University, Ramat Gan, 1997, pp. 83–90.
[12] R. Greiner, St. Ruscheweyh, On the approximation of univalent functions by subordinate polynomials in the unit disk, in: R.V.M. Zahar (Ed.), Approximation and Computation, International Series of Numerical Mathematics, Vol. 119, Birkhäuser, Boston, 1994, pp. 261–271.
[13] O. Lehto, K.I. Virtanen, Quasiconformal Mappings in the Plane, 2nd Edition, Springer, New York, 1973.
[14] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer, Berlin, New York, 1992.
[15] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1970.